Exercise 2.3.5

Evaluate (be careful if $n = m$)

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx \qquad \text{for } n > 0,\ m > 0.$$

Use the trigonometric identity

$$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)].$$

Solution

Assume that $n \neq m$. Use the product-to-sum formula for sine-sine.

$$\begin{aligned}
\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx &= \int_0^L \frac{1}{2}\left[\cos\left(\frac{n\pi x}{L}-\frac{m\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L}+\frac{m\pi x}{L}\right)\right]dx \\
&= \frac{1}{2}\left[\int_0^L \cos\frac{(n-m)\pi x}{L}\,dx - \int_0^L \cos\frac{(n+m)\pi x}{L}\,dx\right] \\
&= \frac{1}{2}\left[\frac{L}{(n-m)\pi}\sin\frac{(n-m)\pi x}{L}\bigg|_{x=0}^{x=L} - \frac{L}{(n+m)\pi}\sin\frac{(n+m)\pi x}{L}\bigg|_{x=0}^{x=L}\right] \\
&= \frac{1}{2}\left\{\frac{L}{(n-m)\pi}[\sin(n-m)\pi - \sin 0] - \frac{L}{(n+m)\pi}[\sin(n+m)\pi - \sin 0]\right\} \\
&= \frac{L}{2\pi}\left[\frac{\sin(n\pi - m\pi)}{n-m} - \frac{\sin(n\pi + m\pi)}{n+m}\right] \\
&= \frac{L}{2\pi}\left[\frac{(n+m)(\sin n\pi\cos m\pi - \sin m\pi\cos n\pi) - (n-m)(\sin n\pi\cos m\pi + \sin m\pi\cos n\pi)}{(n-m)(n+m)}\right] \\
&= \frac{L}{2\pi}\left(\frac{-2n\sin m\pi\cos n\pi + 2m\sin n\pi\cos m\pi}{n^2-m^2}\right) \\
&= \frac{L}{\pi}\left(\frac{m\sin n\pi\cos m\pi - n\sin m\pi\cos n\pi}{n^2-m^2}\right)
\end{aligned}$$

If n and m are integers, then this integral is zero. Assume now that $n = m$.

$$\begin{aligned}
\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx &= \int_0^L \sin^2\frac{n\pi x}{L}\,dx \\
&= \int_0^L \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right)dx \\
&= \frac{1}{2}\left(\int_0^L dx - \int_0^L \cos\frac{2n\pi x}{L}\,dx\right) \\
&= \frac{1}{2}\left(L - \frac{L}{2n\pi}\sin\frac{2n\pi x}{L}\bigg|_{x=0}^{x=L}\right) \\
&= \frac{1}{2}\left[L - \frac{L}{2n\pi}(\sin 2n\pi - \sin 0)\right] \\
&= \frac{L}{2}\left(1 - \frac{\sin 2n\pi}{2n\pi}\right)
\end{aligned}$$

If n is an integer, then this integral is $L/2$.